

Bound entanglement and continuous variables

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We introduce the definition of generic bound entanglement for the case of continuous variables. We provide some examples of bound entangled states for that case, and discuss their physical sense in the context of quantum optics. We rise the question of whether the entanglement of these states is generic. As a byproduct we obtain a new many parameter family of bound entangled states with positive partial transpose. We also point out that the “entanglement witnesses” and positive maps revealing the corresponding bound entanglement can be easily constructed.

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Entanglement is a fascinating property of quantum states evoking fundamental [1,2], as well as practical questions. In the context of information theory, it has been proven to be useful in quantum cryptography [3], quantum dense coding [4], quantum teleportation [5] and quantum computation [6]. In order to make entanglement useful despite of the noise coming from interactions with environment, the idea of noisy entanglement distillation has been introduced [7]. The distillation problem, i.e. the question which states are distillable, has a simple solution for low dimensional quantum systems: two spin- $\frac{1}{2}$ particles, or spin-1 plus spin- $\frac{1}{2}$ systems [8]. In those cases *any* noisy entanglement can be distilled to maximally entangled form: For larger spins the existence bound entanglement (BE) i. e. entanglement which is not distillable has been demonstrated [9]. BE represents the result of nontrivial irreversible process in which entanglement is confined to the physical system. It was shown that there is connection of BE with other very interesting quantum phenomena, called nonlocality without entanglement [10] (see [11] for discussion).

It is not trivial to provide examples of states which are BE. It has been shown [9] that any state which is entangled and at the same time satisfies positive partial transpose (PPT) condition [12] is bound entangled. The existence of PPT entangled states was discussed in [13] and the first explicit examples were provided in [14]. The first systematic procedure of constructing such states, employing unextendible product basis (UPB) was provided in Refs. [15]. In the mathematical literature the first examples of matrices which can be treated as prototypes of PPT entangled states were provided by Choi [16]. Here, we shall use the generalised structure of the Choi matrices to provide the first examples of PPT entangled states for continuous variables.

Let us recall that the PPT separability condition [12], applied to a density matrix ϱ requires that the partial transposed matrix ϱ^{T_B} is still a legitimate state. The matrix ϱ^{T_B} associated with an arbitrary product orthonormal $|i, j\rangle$ basis is defined as:

$$\varrho_{m\mu, n\nu}^{T_B} \equiv \langle m, \mu | \varrho^{T_B} | n, \nu \rangle = \varrho_{m\nu, n\mu}, \quad (1)$$

and the Peres criterion [12] requires that $\varrho^{T_B} \geq 0$ for separable ϱ . This statement is *valid* also for the cases when the state is defined on infinite dimensional Hilbert space.

Although the existence of BE states for finite dimensions has been proved, it has not been known so far whether nontrivial examples of BE states exist in the infinite dimensional case. In fact, main investigations of entanglement in the continuous variables area were performed for pure states resulting in nonlocality effects [18], new versions of teleportation [19], quantum computation [20], quantum error correction [21] and quantum dense coding [22]. For mixed states, however, the PPT condition for continuous variables has been, so far, analysed only in situations, in which it is necessary and sufficient for separability. In particular, it has been shown this is the case for Gaussian states [23,24].

In this paper we discuss bound entanglement for continuous variables. We define the requirement any generic bound entangled state must satisfy in that case. We provide the first examples of nontrivial PPT entangled states, *ergo* BE states for continuous variables. We rise the question how generic they are, and discuss also the problem of physical realization of such states.

Of course, one can simply construct a trivial example. Consider, say $3 \otimes 3$ BE state σ , and the infinitely dimensional Hilbert space \mathcal{H} . Let us define infinitely many “copies” of σ labelled by σ_n , each of which has the matrix elements of the original σ , but in basis $S_n = \{|i, j\rangle\}_{i,j=3n}^{3n+3}$. Let $\{p_i\}_{i=1}^{\infty}$ be a infinite sequence of nonzero probabilities, $\sum_{i=1}^{\infty} p_i = 1$. Then the following state

$$\tilde{\sigma} = \bigoplus_{n=1}^{\infty} p_n \sigma_n, \quad (2)$$

is bound entangled, but it has a trivial form from the continuous variables point of view [25]. Actually, it can be reproduced with arbitrary accuracy performing local transformations on states which are of the $3 \otimes 3$ type. Moreover, they can be produced in a *reversible way*. This follows from the fact that the states σ_n and $\sigma_{n'}$ are *locally* orthogonal [17]. That means that Alice and Bob

can distinguish them using local quantum actions and classical communication (LQCC) only. This can be done in a reversible way as both persons can forget the results of measurements. In effect, there is no entanglement between states belonging to sets of Alice vectors $|\psi_i\rangle_{i,j=3n}^{3n+3}$ and Bob ones $|\psi_i\rangle_{i,j=3n'}^{3n'+3}$ for $n \neq n'$.

Thus, in the case of the states (2) we deal effectively with $3 \otimes 3$ type entanglement only. What does that mean from the formal, and more rigorous point of view? One should ask first what does that mean that a state represents a *generic* $N \otimes N$ type entanglement. The answer to this question can be obtained immediately using the recently introduced definition of Schmidt rank [26] for mixed states. Let us recall the definition:

Definition.- Bipartite density matrix ρ has Schmidt rank K iff (i) for any decomposition of ρ , $\{p_i \geq 0, |\psi_i\rangle\}$ with $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ at least one of vectors $|\psi_i\rangle$ has Schmidt rank K , and (ii) there exists a decomposition of ρ with all vectors $\{|\psi_i\rangle\}$ of Schmidt rank at most K .

Thus it is natural from the physical point of view to say that the state represents generic rank K entanglement iff it has Schmidt rank K . We introduce therefore:

Definition.- A state ρ represents generic *continuous variables* or *infinite Schmidt rank entanglement* iff it is the limit of states ρ_n of Schmidt rank K_n , with $\lim_n K_n = \infty$.

In the following we shall focus on the question of existence of *generic* continuous variables entanglement, which would be at the same time bound entanglement, i.e. entanglement which cannot be distilled. We will construct PPT states for continuous variables and argue that they represent generic infinite Schmidt rank entanglement.

For this aim consider first the state

$$|\Psi\rangle = \sum_{n=1}^{\infty} a_n |n, n\rangle, \quad \|\Psi\|^2 = \sum_{n=1}^{\infty} |a_n|^2 = q < \infty, \quad (3)$$

and the family of states

$$|\Psi_{mn}\rangle = c_m a_n |n, m\rangle + (c_m)^{-1} a_m |m, n\rangle, \quad (4)$$

for $n < m$ with (in general) complex a_n and c_n such that $0 < |c_{n+1}| < |c_n| < 1$. Let us assume that the sum $\sum_{n=1}^{\infty} \sum_{m>n}^{\infty} \|\Psi_{mn}\|^2$ is finite. This can be achieved for example by setting $a_n = a^n$, $c_n = c^n$ for some $0 < a < c < 1$. The sum becomes then a double geometric series and is given by $a^4 c^4 (1 - c^2)^{-1} (1 - a^2 c^2)^{-1} + a^6 (c^2 - a^2)^{-1} (c^2 - a^4)^{-1}$. Under the above assumptions the matrix

$$\varrho = \frac{1}{A} (|\Psi\rangle\langle\Psi| + \sum_{n=1}^{\infty} \sum_{m>n}^{\infty} |\Psi_{mn}\rangle\langle\Psi_{mn}|), \quad (5)$$

with the normalising factor

$$A \equiv \|\Psi\|^2 + \sum_{n=1}^{\infty} \sum_{m>n}^{\infty} \|\Psi_{mn}\|^2,$$

represents a legitimate quantum state in the space $l^2(\mathcal{C}) \otimes l^2(\mathcal{C})$, where $l^2(\mathcal{C})$ is the space of all complex sequences $\{z_n\}$, $\sum_{n=1}^{\infty} |z_n|^2 < \infty$. It can be seen by inspection that the above state satisfies $\varrho = \varrho^{TB}$, and thus has the PPT property. In fact we have chosen $|\Psi_{mn}\rangle$ in Eq. (4) to ensure this property. It follows immediately that pure state entanglement cannot be distilled from (5). For this aim simple arguments from the Ref. [9] can be recalled, and applied to the separable superoperators in infinitely dimensional space.

Subsequently we shall show that the above states are entangled, and, thus being the PPT states represent bound entanglement. To this aim we shall prove that the state (5) has the following property:

Property.- Any local measurement of state ϱ

$$\varrho \rightarrow \varrho' = \frac{P \otimes Q \varrho P \otimes Q}{\text{Tr}(P \otimes Q \varrho P \otimes Q)} \quad (6)$$

by means of P , Q projecting onto the space $\text{span}\{|n_1\rangle, \dots, |n_K\rangle\}$ on Alice and Bob's sides respectively results in $K \otimes K$ bound entangled state ϱ' . From the above property it follows immediately that ϱ is a bound entangled state.

Proof.- Let us first prove that for any K the state ϱ' is a $K \otimes K$ BE state. To this aim we observe that after local filter operation corresponding to the operator $V = \text{diag}[a_{n_1}^{-1}, \dots, a_{n_K}^{-1}]$ on the Alice side, and a suitable unitary transformation $U_1 \otimes U_2$ (that transforms $|n_m\rangle \rightarrow |m\rangle$ on both Alice and Bob's sides), the state becomes proportional to the particularly simple matrix:

$$\Sigma \equiv |\Phi\rangle\langle\Phi| + \sum_{n=1}^K \sum_{m>n}^K |\Phi_{mn}\rangle\langle\Phi_{mn}|, \quad (7)$$

with $|\Phi\rangle = \sum_{n=1}^K |n, n\rangle$, and $|\Phi_{mn}\rangle = \alpha_m |n, m\rangle + \alpha_m^{-1} |m, n\rangle$. In the following, we shall use the general definition of ϱ' as well, but after the above defined local action the state Σ is proportional to the matrix with parameters $\alpha_m = c_{n_m}$. It means in particular that $0 < \alpha_{i+1} < \alpha_i < 1$. We shall prove that Σ does not have any product vector in its range, *ergo* that, if normalised, Σ is an entangled state. That will mean, however, also that the state ϱ' is bound entangled, since the local filtering and the local unitary operations are reversible with nonzero probability.

Suppose, that there was a nonzero product state $|\psi, \phi\rangle$ in the range of the matrix (7). Since the range of Σ is spanned by its eigenvectors, there would exist some g , g_{ij} , $i = 1, \dots, K$, $j > i$ such that:

$$g|\Phi\rangle + \sum_{i=1}^K \sum_{j>i}^K g_{ij} |\Phi_{ij}\rangle = |\psi, \phi\rangle. \quad (8)$$

Suppose first that we would have $g \neq 0$ in (8). Then, we could set $g = 1$, and the following constraints would immediately follow from (8):

$$\begin{aligned}\psi &= [x_1, \dots, x_K], \\ \phi &= [(x_1)^{-1}, \dots, (x_K)^{-1}]\end{aligned}\quad (9)$$

for some numbers $\{x_i\}$ which are all nonzero. Substituting (9) into (8) leads to the equations:

$$\begin{aligned}g_{ij}\alpha_j &= \frac{x_i}{x_j}, \\ g_{ij}(\alpha_j)^{-1} &= \left(\frac{x_i}{x_j}\right)^{-1}, \\ \text{for } i &= 1, \dots, K, \quad j > i.\end{aligned}\quad (10)$$

Since numbers $\{x_i\}$ are nonzero, the coefficients $\{g_{ij}\}$ have to be nonzero too. Thus we have $\alpha_j^2 = (\frac{x_i}{x_j})^2$ for every $i = 1, \dots, K-1, j > i$. We can, however, put $x_1 = 1$, and then we get that all x_i^2 's are equal, and that

$$\alpha_j^2 = 1, \text{ for } j = 2, \dots, K-1. \quad (11)$$

This is in contradiction with the condition $0 < \alpha_{i+1} < \alpha_i < 1$ fulfilled by Σ .

Consider now the case when $g = 0$ holds in equation (8). That would mean, keeping the same notation for ψ , i.e. $|\psi\rangle = [x_1, \dots, x_K]$ that we could have $|\phi\rangle = [y_1, \dots, y_K]$ with $y_i \neq 0$ iff $x_i = 0$. But, if we examine the equation (8) under those conditions we get immediately that all g_{ij} parameters must vanish, so that the whole LHS of the equation becomes then equal to zero. It means that there is no product vector in the range of the matrix Σ . Following previous discussion it is not difficult to see that the same holds for states ϱ' , which are thus (by virtue of the range criterion of Ref. [14]) entangled. Collecting all the above observations, we see that the **Property** of the original matrix ϱ holds. \square

Unfortunately, it is not easy to see that ϱ' (ϱ) represents the generic rank K (∞) entanglement. In fact ϱ' contains in the mixture the pure state of Schmidt rank K which cannot be distinguished from the rest of the mixture in a reversible manner, since its reduced density matrix has full rank K . Certainly, ϱ does not consist of locally orthogonal representation of finite Schmidt rank entanglement. This can be easily seen from the fact that local orthogonality is a stronger property than orthogonality in case of pure states vectors. Had ϱ been a locally orthogonal mixture of finite Schmidt rank state, the eigenvectors of ϱ would have been locally orthogonal, and of finite Schmidt rank, which is obviously not true, since one of the eigenvectors of ϱ is of infinite Schmidt rank. Nevertheless, we have not been able to show, so far, that the Schmidt rank of the proposed states is infinite. It is, however, quite likely that either these states, or some modification of them possess that property.

Finite dimensional bound entangled states .- It is remarkable that as a byproduct we have obtained here a new family of $K \otimes K$ bound entangled states for an arbitrary K . These are the states $\sigma = \frac{\Sigma}{Tr(\Sigma)}$, with Σ violating

one (or more) of the $K-1$ conditions (11). In this notation we recall the Choi matrix as a special case of Σ with $K = 3$, and all α 's equal to 2 (see [16]).

The corresponding "entanglement witnesses" and positive maps .- It should be pointed out that any BE state from the last paragraph (i. e. $\frac{\Sigma}{Tr(\Sigma)}$ violating condition (11)) has no product vector in its range. Thus the projector P onto its range has no product vector in its range as well. This is the same as in the projector orthogonal to UPB set of vectors [15], and thus *mutatis mutandis* the approach from the paper [27] can be immediately applied to reproduce both entanglement witnesses, as well as the corresponding positive maps.

Possibility of physical realization .- Let us now discuss a possibility of physical realization of the states of the type of ϱ , as states of two photon modes of electromagnetic field of equal or similar frequency, and orthogonal polarisations.

Let us set $a_n = e^{-\beta n}$, $c_n = e^{-\gamma n}$, $\gamma < \beta$, and let us denote the corresponding photon creation and annihilation operators of the two modes considered as A^\dagger , A , B^\dagger , B , respectively. The state ϱ can be represented as a mixture

$$\varrho \sim |\Psi\rangle\langle\Psi| + \sum_{k=1}^{\infty} \varrho_k, \quad (12)$$

where

$$\varrho_k = V\delta(B^\dagger B - A^\dagger A - k)V^\dagger, \quad (13)$$

where $V = e^{-\beta A^\dagger A - \gamma B^\dagger B} + Ue^{-(\beta - \gamma)B^\dagger B}$, U is the unitary operator that transforms A photons into B photons, while the operator function $\delta(\cdot)$ is the operator valued Kronecker delta, $\delta(x) = 0$, except for $x = 0$, when $\delta(x) = 1$. We propose the following prescription in order to generate the states corresponding to the subsequent terms in the mixture (12):

i) The state $|\Psi\rangle \sim \exp(-\gamma B^\dagger A)|0, 0\rangle$ can be created as a two mode squeezed state, for instance in the process of degenerate parametric amplification [18,22]. In fact, such states have been used for teleportation with continuous variables [22].

ii) Each of the terms ϱ_k can be obtained by applying the positive operator valued measurement to the states $\delta_k = \delta(B^\dagger B - A^\dagger A - k)$, that transforms (with some probability)

$$\delta(B^\dagger B - A^\dagger A - k) \rightarrow V\delta(B^\dagger B - A^\dagger A - k)V^\dagger.$$

iii) The operator V can be realized by using an ancilla (say a two level atom medium) with levels $|0\rangle$ and $|1\rangle$ that undergoes losses (via spontaneous emission, or ionization) proportional to $-\beta A^\dagger A - \gamma B^\dagger B$ for the level $|0\rangle$ and $-(\beta - \gamma)B^\dagger B$ for $|1\rangle$, and dispersive dynamics governed by the Hamiltonian $h \propto |0\rangle\langle 0| + (A^\dagger B + B^\dagger A)|1\rangle\langle 1|$. One should first prepare the ancilla in the state $(|0\rangle + |1\rangle)$, wait appropriate time so that the dispersive dynamics

will realize the controlled U operation, $|0\rangle\langle 0| + U|1\rangle\langle 1|$, and project the system then onto the initial ancilla state.
iv) The main problem consist thus in generating the states δ_k ,

$$\sum_n \sum_{k=1}^{\infty} |n, n+k\rangle\langle n, n+k|.$$

These states cannot be normalised, but we can always regularize them by including part of the thermal noise operators into their definition. In order to realize the states δ_k , we propose to use a K level ancilla (K -level atom) constituting a Kerr medium with the states $|i\rangle$, $i = 1, \dots, K$. The Hamiltonian should now be $\tilde{h} = \sum_i \Delta_i (A^\dagger A, B^\dagger B) |i\rangle\langle i|$, where the intensity dependent energy shift of the i -th level is $\Delta_i (A^\dagger A, B^\dagger B) = x_i (A^\dagger A - B^\dagger B + k)$. Kerr effect should be here of the opposite sign for the two modes in question, but of the same magnitude. The last term in Δ_i represents normal linear phase shift.

v) The idea is then to prepare the K -level ancilla in a more or less equal weight superposition of the states $|i\rangle$, evolve then the system according to the dynamics governed by \tilde{h} , and project at the end on the initial state of the ancilla. Note that in such case, the action of such "phase shifter" on the state $|n, m\rangle$ will be

$$|n, m\rangle \rightarrow \sum_i e^{ix_i(n-m+k)} |n, m\rangle, \quad (14)$$

which, if K is sufficiently large and x_i cover a broad variety of phases, provides a finite bandwidth approximation of the Kronecker delta.

Summarising, we have presented the first nontrivial example of bound entangled states for continuous variables. We have presented strong evidence that these state represent generic bound state entanglement of infinite rank.

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